Green-Lazarsfeld's Gonality Conjecture for a Generic Curve of Odd Genus

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Abstract

We prove Green-Lazarsfeld's gonality conjecture for generic curves of odd genus. The proof uses, among other things, the main result of [1], and Green's syzygy canonical conjecture for generic curves of odd genus, [12]. The even-genus case was previously solved in the joint work [3].

1   Introduction.

Let C be a smooth, projective, irreducible, complex curve, and L a globally generated line bundle on C, and denote by $K_{p,q}(C,L)$ the Koszul cohomology of C with value in L. By definition, it is the cohomology of the complex:

$$\bigwedge^{p+1} H^0(C,L) \otimes H^0(C,L^{q-1}) \to \bigwedge^{p} H^0(C,L) \otimes H^0(C,L^q) \to \bigwedge^{p-1} H^0(C,L) \otimes H^0(C,L^{q+1}).$$

Many geometric phenomena regarding the image of C in $\mathbb{P}H^0(C,L)^*$, such as projective normality, or being cut out by quadrics can be expressed in terms of vanishing of Koszul cohomology.

There are a number of preliminary results which suggested that other geometric properties can be related to Koszul cohomology. A good example of a possible interaction between geometry and Koszul cohomology is provided by the Nonvanishing Theorem of Green and Lazarsfeld (cf. [5, Appendix]). In a particular case, it shows that curves with special geometry have non-trivial Koszul cohomology groups. More precisely, if we set d the gonality of C, and we suppose that L is of sufficiently large degree, then one has:

$$K_{h^0(C,L)-d-1,1}(C,L) \neq 0.$$

It was conjectured by Green and Lazarsfeld, [7, Conjecture 3.7], that this result were optimal, that is:

$$(1) \qquad\qquad K_{h^0(C,L)-d,1}(C,L) = 0,$$

for all L of sufficiently large degree. Broadly speaking, it would mean that gonality of curves could be read off minimal resolutions of embeddings of sufficiently large degree.

We prove here the following:

Theorem 1.1. *Green-Lazarsfeld's gonality conjecture holds for a generic curve of odd genus.*

The case of generic curves of odd genus is exactly the case left away from the joint work [3], where the gonality conjecture was verified for a generic curve of genus g and gonality d with $\frac{g}{3} + 1 \leqslant d < [\frac{g+3}{2}]$. Recall that curves of given genus g and gonality d are parametrized by an irreducible quasi-projective variety. Beside, we always have $d \leqslant [\frac{g+3}{2}]$, and the maximal gonality is realized on a non-empty open set of moduli space of curves of genus g. Therefore, Theorem 1.1 and the results of [3] complete each other, showing eventually that Green-Lazarsfeld's gonality conjecture holds for a generic curve of genus g and gonality d with $\frac{g}{3} + 1 \leqslant d$.

The techniques used for proving Theorem 1.1 are standard, and very similar to those used in [1], [3], [12], see Section 2 for details. Green's conjecture plays a central role in the whole proof. An alternative proof of the generic gonality conjecture for the even-genus case is sketched in Remark 2.5. In Section 3, we prove a result, Proposition 3.1, which makes a clear difference between the case of generic curves of odd genus, and the curves analysed in [3], justifying the strategy chosen for proving Theorem 1.1.

Notation and conventions.

If V a finite-dimensional complex vector space, and $S(V)$ denotes its symmetric algebra, for any graded $S(V)$–module $B = \bigoplus_{q \in \mathbb{Z}} B_q$ there is a naturally defined complex of vector spaces, called the *Koszul complex* of B (cf. [5, Definition 1.a.7]),

$$... \longrightarrow \bigwedge^{p+1} V \otimes B_{q-1} \xrightarrow{d_{p+1,q-1}} \bigwedge^{p} V \otimes B_q \xrightarrow{d_{p,q}} \bigwedge^{p-1} V \otimes B_{q+1} \longrightarrow ...,$$

where p, and q are integer numbers. The cohomology of this complex is denoted by

$$K_{p,q}(B, V) = \operatorname{Ker} d_{p,q}/\operatorname{Im} d_{p+1,q-1}.$$

In the algebro-geometric context, if X is a complex projective variety, $L \in \operatorname{Pic}(X)$ is a line bundle, $\mathcal{F}$ is a coherent sheaf, and $V = H^0(X, L)$, one usually computes Koszul cohomology for the $S(V)$-module $B = \bigoplus_{q \in \mathbb{Z}} H^0(X, L^q \otimes \mathcal{F})$. The standard notation is $K_{p,q}(X, \mathcal{F}, L) = K_{p,q}(B, V)$. If $\mathcal{F} \cong \mathcal{O}_X$, we drop it, and write simply $K_{p,q}(X, L)$.

Throughout this paper, we shall refer very often to [5] and [6] for the basic facts of the Koszul cohomology theory.

2   Proof of Theorem 1.1.

A generic curve of odd genus $2k+1$ is of gonality $k+2$. Since elliptic curves are fairly well understood, we can suppose $k \geqslant 1$. Recall that vanishing of Koszul cohomology is an open property (see, for example, [4]), and the moduli space of curves of given

genus is irreducible, so it would suffice to exhibit one curve C of odd genus $2k + 1$, and one nonspecial line bundle L_C on C such that

$$K_{h^0(C,L_C)-k-2,1}(C, L_C) = 0.$$

This principle has already been used in [1] and [3], and is based on the main result of [1] (see also [3, Theorem 2.1]):

Theorem 2.1. *If L is a nonspecial line bundle on a curve C, which satisfies $K_{n,1}(C, L) = 0$, for a positive integer n, then, for any effective divisor E of degree e, we have $K_{n+e,1}(C, L + E) = 0$.*

We construct C and L_C in the following way. Let S be a $K3$ surface whose Picard group is freely generated by a very ample line bundle $\mathcal{L}$, and by one smooth rational curve Δ, such that $\mathcal{L}^2 = 4k$, and $\mathcal{L}.\Delta = 3$. Such a surface does exist, as one can see by analysing the period map (see, for example, [10, Lemma 1.2]). Denote $L = \mathcal{L} + \Delta$. Then one can show:

Lemma 2.2. *All smooth curves in the linear systems $|\mathcal{L}|$ and $|L|$ have maximal Clifford index, and thus maximal gonality, too.*

Proof: The proof runs exactly like in [12, Proposition 1], using [8].

$\square$

Let $C \in |\mathcal{L}|$ be a smooth curve, and set $L_C = L_{|C}$. Then C is of genus $g = 2k+1$, and L_C is of degree $2g + 1 = 4k + 3$, thus nonspecial. Beside, the curve C is generic, and $h^0(C, L_C) = 2k + 3$, so the vanishing (1) predicted by the gonality conjecture for the pair (C, L_C) becomes:

(2) $$K_{k+1,1}(C, L_C) = 0.$$

This vanishing will be a consequence of another two Lemmas which are proven below.

Lemma 2.3. $K_{k+1,1}(S, L) = 0$.

Proof: Remark that any smooth curve $D \in |L|$ is of genus $2k + 3$. From [12], [9], and Lemma 2.2, it follows that Green's conjecture is valid for D. Therefore $K_{k+1,1}(D, K_D) = 0$. From Green's hyperplane section theorem [5, Theorem 3.b.7], we conclude $K_{k+1,1}(S, L) = 0$.

$\square$

Lemma 2.4. *For any integer p, we have a natural isomorphism $K_{p,1}(S, L) \cong K_{p,1}(C, L_C)$.*

Proof: We set $D = C + \Delta$; it is a connected reduced divisor on S. All the hypotheses of Green's hyperplane section theorem [5, Theorem 3.b.7] are fulfilled, so one has isomorphisms

$$K_{p,1}(S, L) \cong K_{p,1}(D, L_D),$$

for all integers p, where we denoted $L_D = L_{|D}$.

We use next the natural short exact sequence of coherent sheaves on S (see also [2]):

$$0 \to \mathcal{O}_\Delta(-C) \to \mathcal{O}_D \to \mathcal{O}_C \to 0,$$

which yields furthermore to an exact sequence, for any integer q:

$$0 \to H^0(\Delta, \mathcal{O}_\Delta(L^{q-1} + \Delta)) \to H^0(D, L_D^q) \to H^0(C, L_C^q)$$

We analyse the maps $H^0(D, L_D^q) \to H^0(C, L_C^q)$ for different q. If $q = 0$ the corresponding map is obviously an isomorphism. For $q = 1$, we also have an isomorphism $H^0(D, L_D) \cong H^0(C, L_C)$, since $\mathcal{O}_\Delta(\Delta) \cong K_\Delta \cong \mathcal{O}_{\mathbb{P}^1}(-2)$. Moreover, since $(L + \Delta).\Delta < 0$, for $q = 2$, we obtain an inclusion $H^0(D, L_D^2) \subset H^0(C, L_C^2)$. These facts reflect into an isomorphism between Koszul cohomology groups $K_{p,1}(D, L_D) \cong K_{p,1}(C, L_C)$ for any integer p (apply, for example, [1, Remark 1.1]). Therefore, for any p, $K_{p,1}(S, L)$ is isomorphic to $K_{p,1}(C, L_C)$.

□

Summing up, $K_{k+1,1}(C, L_C)$ is isomorphic to $K_{k+1,1}(S, L)$, and the latter vanishes, which proves (2), and Theorem 1.1, too.

□

Remark 2.5. A similar idea can be used to give an alternative proof of the gonality conjecture for a generic curve of even genus. Let us sketch the proof in a few words.

Let S be a $K3$ surface whose Picard group is freely generated by a very ample line bundle $\mathcal{L}$, and by one smooth rational curve Δ, such that $\mathcal{L}^2 = 4k - 2$, and $\mathcal{L}.\Delta = 2$, and let furthermore $C \in |\mathcal{L}|$ be a smooth curve. Denote $L = \mathcal{L} + \Delta$, and $L_C = L_{|C}$. Then C is of genus $g = 2k$ and gonality $k + 1$, and L_C is of degree $2g$.

Since $h^0(C, L_C) = 2k + 1$, the vanishing (1) predicted by the gonality conjecture for the pair (C, L_C) becomes $K_{k,1}(C, L_C) = 0$.

A smooth curve in the linear system $|L|$ has genus $2k + 1$ and gonality $k + 2$, and thus it satisfies Green's conjecture, [12]. It implies $K_{k,1}(S, L) = 0$, after having applied Green's hyperplane section theorem.

The proof of Lemma 2.4 can be easily addapted to obtain an isomorphism $K_{p,1}(S, L) \cong K_{p,1}(C, L_C)$, for any p. In particular, for $p = k$, we obtain $K_{k,1}(C, L_C) = 0$.

Remark 2.6. It follows from Theorem 1.1 and Theorem 2.1 that the vanishing (1) is valid for any line bundle of degree at least $3g + 1$ over a generic curve of odd genus g. In the even-genus case, the lower bound was $3g$, see [3]. We could predict thus that, up to one unit, the lower bound $3g$, indicated by Green and Lazarsfeld as one of the possible replacements for the more ambiguous "*of sufficiently large degree*", is correct.

3 Why adding three points to the canonical bundle?

The strategy used in [3] to verify the gonality conjecture for a curve C of genus g, and gonality d with $\frac{g}{3} + 1 \leqslant d < [\frac{g+3}{2}]$ was to look at bundles of type $K_C + x + y$, where x, and y were suitably chosen points of C (see also Remark 2.5). For Theorem 1.1 instead, we needed to add three points to the canonical bundle in order to have the desired vanishing (1). This choice is justified by the following remark, which shows that on a generic curve C of odd genus, bundles of type $K_C + x + y$ never verify (1).

Proposition 3.1. *Let C be a curve of odd genus $2k+1 \geqslant 3$, and maximal gonality $k+2$. Then, for any two points x, and y of C, the dimension of $K_{k,1}(C, K_C + x + y)$ equals the binomial coefficient $\begin{pmatrix} 2k+1 \\ k+2 \end{pmatrix}$.*

For the proof of Proposition 3.1, we need the following elementary Lemma.

Lemma 3.2. *Let X be an irreducible projective manifold, and $D \neq 0$ be an effective divisor. Then, for any $L \in \mathrm{Pic}(X)$, and any integer $p \geqslant 1$, we have an exact sequence:*

$$0 \to \bigwedge^{p+1} H^0(X, L - D) \to K_{p,1}(X, -D, L) \to K_{p,1}(X, L).$$

Proof: Let $V = H^0(X, L)$, and consider the graded $S(V)$–modules

$$A = \bigoplus_{q \in \mathbb{Z}} H^0(X, L^q - D), \ B = \bigoplus_{q \in \mathbb{Z}} H^0(X, L^q),$$

and $C = B/A$, where the inclusion of A in B is given by the multiplication with the non-zero section of $\mathcal{O}_X(D)$ vanishing along D. Obviously, $A_0 = 0$, and $C_0 \cong \mathbb{C}$. The long cohomology sequence for syzygies yields to an exact sequence:

$$0 \to \mathrm{Ker}\left(\bigwedge^{p+1} V \to \bigwedge^p V \otimes C_1\right) \to K_{p,1}(X, -D, L) \to K_{p,1}(X, L) \to \dots$$

We aim to prove that

$$\mathrm{Ker}\left(\bigwedge^{p+1} V \to \bigwedge^p V \otimes C_1\right) \cong \bigwedge^{p+1} H^0(X, L - D).$$

For this, choose a basis $\{w_1, ..., w_N\} \subset V$, such that $\{w_1, ..., w_s\}$ is a basis of $H^0(X, L - D)$, and pick an element $\alpha = \sum_{1 \leqslant i_1 < ... < i_{p+1} \leqslant N} \alpha_{i_1 ... i_{p+1}} w_{i_1} \wedge ... \wedge w_{i_{p+1}} \in \bigwedge^{p+1} V$. It belongs to $\mathrm{Ker}\left(\bigwedge^{p+1} V \to \bigwedge^p V \otimes C_1\right)$ if and only if the following relations are satisfied, for any $1 \leqslant k_1 < ... < k_p \leqslant N$:

$$\sum_{k \notin \{k_1,\dots,k_p\}} (-1)^{\#\{k_i < k\}} \alpha_{k_1 \dots k \dots k_p} w_k \in H^0(X, L - D).$$

In particular, for any $1 \leqslant k_1 < \dots < k_p \leqslant N$, and $k > s$, $\alpha_{k_1 \dots k \dots k_p} = 0$, in other words all $\alpha_{i_1 \dots i_{p+1}}$ with $i_{p+1} > s$ vanish, so α belongs to $\bigwedge^{p+1} H^0(X, L - D)$.

$\square$

Proof of Proposition 3.1: We compute first the dimension of $K_{k-1,2}(C, K_C + x + y)$. By Green's duality Theorem it equals the dimension of $K_{k+1,1}(C, -x - y, K_C + x + y)$. We know $K_{k,1}(C, K_C) = 0$, as C satisfies Green's conjecture. From [1, Theorem 3], it follows $K_{k+1,1}(C, K_C + x + y) = 0$. Then from Lemma 3.2 we obtain an isomorphism

$$\bigwedge^{k+2} H^0(C, K_C) \cong K_{k+1,1}(C, -x - y, K_C + x + y).$$

Therefore, the dimension of $K_{k+1,1}(C, -x-y, K_C+x+y)$, and thus of $K_{k-1,2}(C, K_C + x + y)$, equals the binomial coefficient $\begin{pmatrix} 2k+1 \\ k+2 \end{pmatrix}$. What is left from the proof is a combinatorial computation. Analysing the Koszul complex which computes $K_{k,1}(C, K_C + x + y)$, one can prove:

Lemma 3.3. *The Euler characteristic of the complex*

$$0 \to \bigwedge^{k+1} H^0(K_C + x + y) \to \bigwedge^{k} H^0(K_C + x + y) \otimes H^0(K_C + x + y) \to \dots$$

equals zero.

Proof: Standard combinatorics. Use $h^0(C, K_C^q + qx + qy) = 4qk + 2q - 2k$, for all $q \geqslant 1$.

$\square$

Knowing that $K_{k-j+1,j}(C, K_C + x + y)$ vanishes for all $j \neq 1, 2$, we conclude that $K_{k,1}(C, K_C + x + y)$ and $K_{k-1,2}(C, K_C + x + y)$ have the same dimension.

$\square$

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